

# ON THE ANDREWS-STANLEY REFINEMENT OF RAMANUJAN'S PARTITION CONGRUENCE MODULO 5

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ABSTRACT. In a recent study of sign-balanced, labelled posets Stanley [13], introduced a new integral partition statistic

$$\text{srank}(\pi) = \mathcal{O}(\pi) - \mathcal{O}(\pi'),$$

where  $\mathcal{O}(\pi)$  denotes the number of odd parts of the partition  $\pi$  and  $\pi'$  is the conjugate of  $\pi$ . In [1] Andrews proved the following refinement of Ramanujan's partition congruence mod 5:

$$p_0(5n+4) \equiv p_2(5n+4) \equiv 0 \pmod{5},$$

$$p(n) = p_0(n) + p_2(n),$$

where  $p_i(n)$  ( $i = 0, 2$ ) denotes the number of partitions of  $n$  with  $\text{srank} \equiv i \pmod{4}$  and  $p(n)$  is the number of unrestricted partitions of  $n$ . Andrews asked for a partition statistic that would divide the partitions enumerated by  $p_i(5n+4)$  ( $i = 0, 2$ ) into five equinumerous classes.

In this paper we discuss two such statistics. The first one, while new, is intimately related to the Andrews-Garvan [2] crank. The second one is in terms of the 5-core crank, introduced by Garvan, Kim and Stanton [9]. Finally, we discuss some new formulas for partitions that are 5-cores.

## 1. INTRODUCTION

Let  $p(n)$  be the number of unrestricted partitions of  $n$ . Ramanujan discovered and later proved that

$$(1.1) \quad p(5n+4) \equiv 0 \pmod{5},$$

$$(1.2) \quad p(7n+5) \equiv 0 \pmod{7},$$

$$(1.3) \quad p(11n+6) \equiv 0 \pmod{11}.$$

Dyson [5] was the first to consider combinatorial explanations of these congruences. He defined the rank of a partition as *the largest part minus the number of parts* and made the empirical observations that

$$(1.4) \quad N(k, 5, 5n+4) = \frac{p(5n+4)}{5}, \quad 0 \leq k \leq 4,$$

$$(1.5) \quad N(k, 7, 7n+5) = \frac{p(7n+5)}{7}, \quad 0 \leq k \leq 6,$$

where  $N(k, m, n)$  denotes the number of partitions of  $n$  with rank congruent to  $k$  modulo  $m$ . Equation (1.4) means that the residue of the rank mod 5 divides the partitions of  $5n+4$  into five equal classes. Similarly, (1.5) implies that the residue of

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the rank mod 7 divides the partitions of  $7n+5$  into seven equal classes. Dyson's rank failed to explain (1.3), and so Dyson conjectured the existence of a hypothetical statistic, called the crank, that would explain the Ramanujan congruence mod 11. Identities (1.4)-(1.5) were later proved by Atkin and Swinnerton-Dyer [3]. Andrews and Garvan [2] found a crank for all three Ramanujan congruences (1.1)-(1.3). Their crank is defined as follows

$$(1.6) \quad \text{crank}(\pi) = \begin{cases} \ell(\pi), & \text{if } \mu(\pi) = 0, \\ \tilde{\nu}(\pi) - \mu(\pi), & \text{if } \mu(\pi) > 0, \end{cases}$$

where  $\ell(\pi)$  denotes the largest part of  $\pi$ ,  $\mu(\pi)$  denotes the number of ones in  $\pi$  and  $\tilde{\nu}(\pi)$  denotes the number of parts of  $\pi$  larger than  $\mu(\pi)$ .

Later, Garvan, Kim and Stanton [9] found different cranks, which also explained all three congruences (1.1)-(1.3). Their approach made essential use of  $t$ -cores of partitions and led to explicit bijections between various equinumerous classes. In particular, they provided what amounts to a combinatorial proof of the formula

$$(1.7) \quad \sum_{n \geq 0} p(5n+4)q^n = 5 \prod_{m \geq 1} \frac{(1-q^{5m})^5}{(1-q^m)^6},$$

considered by Hardy to be an example of Ramanujan's best work.

The main results of [2] can be summarized as

$$(1.8) \quad M(k, 5, 5n+4) = \frac{p(5n+4)}{5}, \quad 0 \leq k \leq 4,$$

$$(1.9) \quad M(k, 7, 7n+5) = \frac{p(7n+5)}{7}, \quad 0 \leq k \leq 6,$$

$$(1.10) \quad M(k, 11, 11n+6) = \frac{p(11n+6)}{11}, \quad 0 \leq k \leq 10,$$

and

$$(1.11) \quad 1 + (x + x^{-1} - 1)q + \sum_{n \geq 1} \sum_m \tilde{M}(m, n) x^m q^n = \prod_{n \geq 1} \frac{(1 - q^n)}{(1 - xq^n)(1 - x^{-1}q^n)},$$

where  $\tilde{M}(m, n)$  denotes the number of partitions of  $n$  with crank  $m$  and  $M(k, m, n)$  denotes the number of partitions of  $n$  with crank congruent to  $k$  modulo  $m$ .

In [7] Garvan found a refinement of (1.1)

$$(1.12) \quad M(k, 2, 5n+4) \equiv 0 \pmod{5}, \quad k = 0, 1$$

together with the combinatorial interpretation

$$(1.13) \quad M(2k + \alpha, 10, 5n+4) = \frac{M(\alpha, 2, 5n+4)}{5}, \quad 0 \leq k \leq 4,$$

with  $\alpha = 0, 1$ .

Recently, a very different refinement of (1.1) was given by Andrews [1]. Building on the work of Stanley [13], Andrews examined partitions  $\pi$  classified according to  $\mathcal{O}(\pi)$  and  $\mathcal{O}(\pi')$ , where  $\mathcal{O}(\pi)$  denotes the number of odd parts of the

partition  $\pi$  and  $\pi'$  is the conjugate of  $\pi$ . He used recursive relations to show that

$$(1.14) \quad G(z, y, q) := \sum_{n, r, s \geq 0} S(n, r, s) q^n z^r y^s = \frac{(-zyq; q^2)_\infty}{(q^4; q^4)_\infty (z^2 q^2; q^4)_\infty (y^2 q^2; q^4)_\infty},$$

where  $S(n, r, s)$  denotes the number of partitions  $\pi$  of  $n$  with  $\mathcal{O}(\pi) = r$ ,  $\mathcal{O}(\pi') = s$ , and

$$(1.15) \quad (a; q)_\infty = \lim_{n \rightarrow \infty} (a; q)_n,$$

$$(1.16) \quad (a; q)_n = (a)_n = \begin{cases} 1, & \text{if } n = 0, \\ \prod_{j=0}^{n-1} (1 - aq^j), & \text{if } n > 0. \end{cases}$$

A direct combinatorial proof of (1.14) was later given by A. Sills [12], A. J. Yee [14] and C. Boulet [4]. Actually, C. Boulet proved a stronger version of (1.14) with one extra parameter. We define the Stanley rank of a partition  $\pi$  as

$$(1.17) \quad \text{srnk}(\pi) = \mathcal{O}(\pi) - \mathcal{O}(\pi').$$

It is easy to see that

$$(1.18) \quad \text{srnk}(\pi) \equiv 0 \pmod{2},$$

so that

$$(1.19) \quad p(n) = p_0(n) + p_2(n),$$

where  $p_i(n)$  ( $i = 0, 2$ ) denotes the number of partitions of  $n$  with  $\text{srnk} \equiv i \pmod{4}$ . We note that (1.14) with  $z = y^{-1} = \sqrt{-1}$  immediately implies the Stanley formula [13, p.8]

$$(1.20) \quad \sum_{n \geq 0} (p_0(n) - p_2(n)) q^n = \frac{(-q; q^2)_\infty}{(q^4; q^4)_\infty (-q^2; q^4)_\infty^2}.$$

Using (1.1), (1.19) and (1.20), Andrews proved the following refinement of (1.1)

$$(1.21) \quad p_0(5n + 4) \equiv p_2(5n + 4) \equiv 0 \pmod{5}.$$

His proof of (1.21) was analytic and so at the end of [1] he posed the problem of finding a partition statistic that would give a combinatorial interpretation of (1.21). The object of this paper is to provide a solution to the Andrews problem. It turns out that there are two distinct integral partition statistics, whose residue mod 5 split the partitions enumerated by  $p_i(5n + 4)$  (with  $i = 0, 2$ ) into five equal classes. The first statistic, which we call the *stcrnk*, is new. However, it is intimately related to the Andrews-Garvan crank (1.6). Unexpectedly, the second statistic is the “5-core crank”, introduced by Garvan, Kim and Stanton [9]. This second statistic not only provides the desired combinatorial interpretation, but it also provides a direct combinatorial proof of (1.21).

The rest of this paper is organized as follows. In Section 2 we define the *stcrnk* and show that is indeed, a statistic asked for in [1]. In Section 3 we briefly review the development in [9]. In Section 4 we state a number of new formulas for partitions that are 5-cores. We sketch the “5-core crank” proof of (1.21) and we conclude with some open problems.

## 2. THE STCRANK

We begin with some preliminaries about partitions and their conjugates. A partition  $\pi$  is a nonincreasing sequence

$$(2.1) \quad \pi = (\lambda_1, \lambda_2, \lambda_3, \dots)$$

of nonnegative integers (parts)

$$(2.2) \quad \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$$

The weight of  $\pi$ , denoted by  $|\pi|$  is the sum of parts

$$(2.3) \quad |\pi| = \lambda_1 + \lambda_2 + \lambda_3 + \dots$$

If  $|\pi| = n$ , then we say that  $\pi$  is a partition of  $n$ . Often it is convenient to use another notation for  $\pi$

$$(2.4) \quad \pi = (1^{f_1}, 2^{f_2}, 3^{f_3}, \dots),$$

which indicates the number of times each integer occurs as a part. The number  $f_i = f_i(\pi)$  is called the frequency of  $i$  in  $\pi$ . The conjugate of  $\pi$  is the partition  $\pi' = (\lambda'_1, \lambda'_2, \lambda'_3, \dots)$  with

$$(2.5) \quad \begin{aligned} \lambda'_1 &= f_1 + f_2 + f_3 + f_4 + \dots \\ \lambda'_2 &= f_2 + f_3 + f_4 + \dots \\ \lambda'_3 &= f_3 + f_4 + \dots \\ &\vdots \end{aligned}$$

Next, we discuss two bijections. The first one relates  $\pi$  and bipartitions  $(\pi_1, \pi_2)$ , where  $\pi_2$  is a partition with no repeated even parts.

**Bijection 1**

$$\pi \xrightarrow{1} (\pi_1, \pi_2),$$

where

$$\begin{aligned} \pi &= (1^{f_1}, 2^{f_2}, 3^{f_3}, \dots), \\ \pi_1 &= (1^{\lfloor f_2/2 \rfloor}, 2^{\lfloor f_4/2 \rfloor}, 3^{\lfloor f_6/2 \rfloor}, \dots), \\ \pi_2 &= (1^{f_1}, 2^{\{f_2\}}, 3^{f_3}, 4^{\{f_4\}}, \dots), \end{aligned}$$

$\lfloor x \rfloor$  is the largest integer  $\leq x$ , and

$$\{x\} = x - 2\lfloor x/2 \rfloor.$$

Indeed, remove from  $\pi$  the maximum even number of even parts. The resulting partition is  $\pi_2$ . The removed even parts can be organized into a new partition  $(2^{\lfloor f_2/2 \rfloor}, 4^{\lfloor f_4/2 \rfloor}, 6^{\lfloor f_6/2 \rfloor}, \dots)$ , which can easily be mapped onto  $\pi_1$ . Clearly, we have

$$(2.6) \quad |\pi| = 4|\pi_1| + |\pi_2|,$$

$$(2.7) \quad \text{srank}(\pi) = \text{srank}(\pi_2),$$

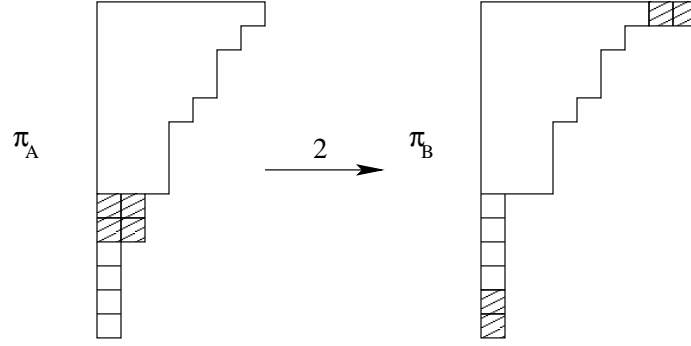


FIGURE 1. Graphical illustration of Bijection 2

so that

$$\begin{aligned}
 \sum_{\pi} q^{|\pi|} y^{\text{srank}(\pi)} &= \sum_{\pi_1} q^{4|\pi_1|} \sum_{\pi_2} q^{|\pi_2|} y^{\text{srank}(\pi_2)} \\
 (2.8) \qquad \qquad \qquad &= \frac{1}{(q^4; q^4)_{\infty}} \sum_{\pi_2} q^{|\pi_2|} y^{\text{srank}(\pi_2)}.
 \end{aligned}$$

Comparing (2.8) and (1.14) with  $zy = 1$ , we see that

$$(2.9) \qquad \sum_{\pi_2} q^{|\pi_2|} y^{\text{srank}(\pi_2)} = \frac{(-q; q^2)_{\infty}}{(y^2 q^2; q^4)_{\infty} (q^2/y^2; q^4)_{\infty}},$$

where the sum is over all partitions with no repeated even parts.

To describe our second bijection we require a few definitions. We say that  $\pi_A$  is a partition of type A iff  $\pi_A \xrightarrow{1} ((1), \pi_2)$ . We say that  $\pi_B = (\lambda_1, \lambda_2, \lambda_3, \dots)$  is a partition of type B iff either  $|\pi_B| \neq 4$ ,  $\lambda_1 - \lambda_2 \geq 2$ ,  $\lambda'_1 - \lambda'_2 \geq 2$ ,  $\lambda_1 - 2$  and  $\lambda_2$  are not identical even integers and  $\pi_B$  has no repeated even parts, or  $\pi_B = (3, 1)$ . Obviously,  $\pi_B \xrightarrow{1} ((0), \pi_B)$ . Our second bijection relates partitions of type A and B.

### Bijection 2

$$\pi_A \xrightarrow{2} \pi_B,$$

where

$$\pi_A = (1^{f_1}, 2^{f_2}, 3^{f_3}, \dots, m^{f_m}),$$

$$\pi_B = \begin{cases} (1^{f_1+2}, 2^{f_2-2}, 3^{f_3}, 4^{f_4}, \dots, (m-1)^{f_{m-1}}, m^{f_m-1}, (m+2)^1), & \text{if } m > 2, \\ (1^{f_1+2}, 4^1), & \text{if } m = 2, f_2 = 3, \\ (1^{f_1+1}, 3^1), & \text{if } m = 2, f_2 = 2, \end{cases}$$

$m \geq 2$ ,  $f_2 = 2, 3$ , and  $f_{2i} = 0, 1$  for  $i > 1$ .

Clearly, we have

$$(2.10) \qquad |\pi_A| = |\pi_B|,$$

$$(2.11) \qquad \text{srank}(\pi_A) = \text{srank}(\pi_B).$$

Next, we define a new partition statistic

$$(2.12) \qquad \text{stcrank}(\pi) = \text{crank}(\pi_1) + \frac{1}{2} \text{srank}(\pi) + \Psi(\pi),$$

where  $\pi_1$  is determined by  $\pi \xrightarrow{1} (\pi_1, \pi_2)$ , and the correction term  $\Psi(\pi) = 1$  if  $\pi$  is of type B and zero, otherwise. We note that

$$(2.13) \quad \text{stcrank}(\pi_A) = -1 + \frac{1}{2}\text{srnk}(\pi_A),$$

and

$$(2.14) \quad \text{stcrank}(\pi_B) = 1 + \frac{1}{2}\text{srnk}(\pi_B).$$

Equipped with the definitions above, we can now prove the following lemma.

**Lemma 2.1.** *If*

$$g(x, y, q) := \sum_{\pi} q^{|\pi|} x^{\text{stcrank}(\pi)} y^{\text{srnk}(\pi)},$$

*then  $g(x, y, q)$  has the product representation*

$$g(x, y, q) = \frac{(q^4; q^4)_{\infty} (-q; q^2)_{\infty}}{(q^4 x, q^4/x, q^2 y^2 x, q^2/(y^2 x); q^4)_{\infty}},$$

*where*

$$(a_1, a_2, a_3, \dots; q)_{\infty} = (a_1; q)_{\infty} (a_2; q)_{\infty} (a_3; q)_{\infty} \cdots$$

*Proof.* If  $\pi$  is not of type B and  $\pi \xrightarrow{1} (\pi_1, \pi_2)$ , then using (2.6)-(2.7), (2.12) we find that

$$(2.15) \quad q^{|\pi|} x^{\text{stcrank}(\pi)} y^{\text{srnk}(\pi)} = q^{4|\pi_1| + |\pi_2|} x^{\text{crank}(\pi_1)} (xy^2)^{\text{srnk}(\pi_2)/2}.$$

On the other hand, if  $\pi = \pi_A$  and  $\pi_A \xrightarrow{2} \pi_B$ , then

$$(2.16) \quad \begin{aligned} & q^{|\pi_A|} x^{\text{stcrank}(\pi_A)} y^{\text{srnk}(\pi_A)} + q^{|\pi_B|} x^{\text{stcrank}(\pi_B)} y^{\text{srnk}(\pi_B)} \\ &= q^{|\pi_A|} (x + x^{-1} - 1) (xy^2)^{\text{srnk}(\pi_A)/2} + q^{|\pi_B|} x^0 (xy^2)^{\text{srnk}(\pi_B)/2}. \end{aligned}$$

Here we have used (2.6)-(2.7) and (2.10)-(2.14).

Equations (2.15), (2.16) imply that

$$(2.17) \quad \sum_{\pi} q^{|\pi|} x^{\text{stcrank}(\pi)} y^{\text{srnk}(\pi)} = \sum_{\pi_1} q^{4|\pi_1|} w(x, \pi_1) \sum_{\pi_2} q^{|\pi_2|} (xy^2)^{\text{srnk}(\pi_2)/2}$$

where

$$(2.18) \quad w(x, \pi_1) = \begin{cases} x + x^{-1} - 1, & \text{if } \pi_1 = (1), \\ x^{\text{crank}(\pi_1)}, & \text{otherwise.} \end{cases}$$

We note that in the first sum on the right side of (2.17) the summation is over unrestricted partitions  $\pi_1$ , and in the second sum the summation is over partitions  $\pi_2$  with no repeated even parts. Finally, recalling (1.11) with  $q \rightarrow q^4$  and (2.9) with  $y^2 \rightarrow xy^2$ , we obtain

$$\sum_{\pi} q^{|\pi|} x^{\text{stcrank}(\pi)} y^{\text{srnk}(\pi)} = \frac{(q^4; q^4)_{\infty}}{(xq^4, q^4/x, q^4)_{\infty}} \cdot \frac{(-q; q^2)_{\infty}}{(xy^2 q^2, q^2/(xy^2); q^4)_{\infty}},$$

as desired.  $\square$

Next we show that

$$(2.19) \quad \text{the coefficient of } q^{5n+4} \text{ in } g(\xi, 1, q) = 0,$$

$$(2.20) \quad \text{the coefficient of } q^{5n+4} \text{ in } g(\xi, \sqrt{-1}, q) = 0,$$

where  $\xi$  is a primitive fifth root of unity ( $\xi^5 = 1$ ). We use the method of [6]. We need Jabobi's triple product identity

$$(2.21) \quad \sum_{n=-\infty}^{\infty} z^n q^{n^2} = (q^2, -qz - q/z; q^2)_{\infty},$$

which implies that

$$(2.22) \quad (q^4; q^4)_{\infty} (-q; q^2)_{\infty} = (q^4, -q^3, -q; q^4)_{\infty} = \sum_{n=-\infty}^{\infty} q^{2n^2+n} = \sum_{k \geq 0} q^{T_k},$$

and

$$(2.23) \quad (q^2 \xi^2, q^2/\xi^2, q^2; q^2)_{\infty} = \frac{1}{1-\xi^2} \sum_{m \geq 0} (-1)^m q^{2T_m} \xi^{-2m} (1 - \xi^{4m+2}).$$

Here  $T_k = k(k+1)/2$ . By Lemma 2.1 and equations (2.22) and (2.23) we have

$$(2.24) \quad \begin{aligned} g(\xi, 1, q) &= \frac{\sum_{k \geq 0} q^{T_k}}{(q^4 \xi, q^4/\xi, q^2 \xi, q^2/\xi; q^4)_{\infty}} = \frac{(q^2 \xi^2, q^2/\xi^2, q^2; q^2)_{\infty}}{(q^{10}; q^{10})_{\infty}} \sum_{k \geq 0} q^{T_k} \\ &= \frac{1}{1-\xi^2} \frac{1}{(q^{10}; q^{10})_{\infty}} \sum_{k, m \geq 0} (-1)^m q^{2T_m + T_k} \xi^{-2m} (1 - \xi^{4m+2}). \end{aligned}$$

Note that  $2T_m + T_k \equiv 4 \pmod{5}$  iff  $k \equiv m \equiv 2 \pmod{5}$ , but then  $1 - \xi^{4m+2} = 0$ . This proves (2.19). The proof of (2.20) is analogous.

Let  $P_i(k, m, n)$  denote the number of partitions of  $n$  with  $\text{srnk} \equiv i \pmod{4}$  and  $\text{stcrnk} \equiv k \pmod{m}$ . Clearly,

$$(2.25) \quad \sum_{k=0}^4 \xi^k \sum_{n \geq 0} P_0(k, 5, n) q^n = \frac{g(\xi, 1, q) + g(\xi, \sqrt{-1}, q)}{2},$$

$$(2.26) \quad \sum_{k=0}^4 \xi^k \sum_{n \geq 0} P_2(k, 5, n) q^n = \frac{g(\xi, 1, q) - g(\xi, \sqrt{-1}, q)}{2},$$

Combining (2.19)-(2.20) and (2.25)-(2.26) we find that

$$(2.27) \quad \sum_{k=0}^4 \xi^k P_i(k, 5, 5n+4) = 0, \text{ (for } i = 0, 2),$$

which implies that

$$(2.28) \quad P_i(0, 5, 5n+4) = P_i(1, 5, 5n+4) = \cdots = P_i(4, 5, 5n+4).$$

On the other hand

$$(2.29) \quad p_i(5n+4) = \sum_{k=0}^4 P_i(k, 5, 5n+4),$$

so that

$$(2.30) \quad P_i(k, 5, 5n+4) = \frac{1}{5} p_i(5n+4),$$

for  $i = 0, 2$  and  $k = 0, 1, 2, 3, 4$ . Thus, we have proved the main result of this section.

**Theorem 2.2.** *The residue of the partition statistic  $\text{stcrank}$  mod 5 divides the partitions enumerated by  $p_i(5n+4)$  with  $i = 0, 2$  into five equinumerous classes.*

We illustrate this theorem in Table 1 below for the 30 partitions of 9. These partitions are organized into five classes with six members each. In each class the first 4 members have  $\text{srnk} \equiv 0 \pmod{4}$  and the remaining two members have  $\text{srnk} \equiv 2 \pmod{4}$ .

TABLE 1.

	$\text{stcrank} \equiv 0 \pmod{5}$	$1 \pmod{5}$	$2 \pmod{5}$	$3 \pmod{5}$	$4 \pmod{5}$
$\text{srnk} \equiv 0 \pmod{4}$	$(3^3)$ $(1^3, 2^1, 4^1)$ $(1^1, 3^1, 5^1)$ $(4^1, 5^1)$	$(1^5, 2^2)$ $(1^4, 5^1)$ $(1^2, 2^1, 5^1)$ $(9^1)$	$(1^4, 2^1, 3^1)$ $(1^3, 3^2)$ $(1^1, 4^2)$ $(2^2, 5^1)$	$(1^1, 2^4)$ $(1^6, 3^1)$ $(1^1, 2^1, 6^1)$ $(2^1, 7^1)$	$(1^9)$ $(1^2, 2^2, 3^1)$ $(2^3, 3^1)$ $(1^2, 7^1)$
$\text{srnk} \equiv 2 \pmod{4}$	$(1^3, 2^3)$ $(1^3, 6^1)$	$(1^1, 2^1, 3^2)$ $(1^2, 3^1, 4^1)$	$(1^5, 4^1)$ $(1^1, 8^1)$	$(1^7, 2^1)$ $(1^1, 2^2, 4^1)$	$(2^1, 3^1, 4^1)$ $(3^1, 6^1)$

Finally, we note that the equation

$$(2.31) \quad \text{srnk}(\pi) = -\text{srnk}(\pi')$$

implies that a partition  $\pi$  is self-conjugate only if  $\text{srnk}(\pi) = 0$ . This means that the involution  $\pi \rightarrow \pi'$  has no fixed points if  $\text{srnk}(\pi) \equiv 2 \pmod{4}$ . Hence,  $2 \mid p_2(5n+4)$  and by (1.21) we have the stronger congruence

$$p_2(5n+4) \equiv 0 \pmod{10}.$$

### 3. $t$ -CORES

In this section we recall some basic facts about  $t$ -cores and briefly review the development in [9]. A partition  $\pi$  is called a  $t$ -core, if it has no rim hooks of length  $t$  [10]. We let  $a_t(n)$  denote the number of partitions of  $n$  which are  $t$ -cores. In what follows,  $\pi_{t\text{-core}}$  denotes a  $t$ -core partition. Given the diagram of a partition  $\pi$  we label a cell in the  $i$ -th row and  $j$ -th column by the least nonnegative integer congruent to  $j-i \pmod{t}$ . The resulting diagram is called a  $t$ -residue diagram [10, p.84].

Let  $P$  be the set of all partitions and  $P_{t\text{-core}}$  be the set of all  $t$ -cores. There is well-known bijection which goes back to Littlewood [11].  $\phi_1 : P \rightarrow P_{t\text{-core}} \times P \times \cdots \times P$ ,

$$(3.1) \quad \phi_1(\pi) = (\pi_{t\text{-core}}, \vec{\hat{\pi}}_t),$$

$$(3.2) \quad \vec{\hat{\pi}}_t = (\hat{\pi}_0, \hat{\pi}_1, \hat{\pi}_2, \dots, \hat{\pi}_{t-1}),$$

such that

$$(3.3) \quad |\pi| = |\pi_{t\text{-core}}| + t \sum_{i=0}^{t-1} |\hat{\pi}_i|.$$



This bijection is described in more detail in [10], [9] and [8]. The following identity is an immediate corollary of this bijection.

$$(3.4) \quad \frac{1}{(q)_\infty} = \sum_{n \geq 0} p(n)q^n = \frac{1}{(q^t; q^t)_\infty} \sum_{n \geq 0} a_t(n)q^n.$$

It can be rewritten as

$$(3.5) \quad \sum_{n \geq 0} a_t(n)q^n = \frac{(q^t; q^t)_\infty}{(q)_\infty}.$$

There is another bijection  $\phi_2$ , introduced in [9]. It is for  $t$ -cores only.  $\phi_2 : P_{t\text{-core}} \rightarrow \{\vec{n} = (n_0, n_1, \dots, n_{t-1}) : n_i \in \mathbb{Z}, n_0 + \dots + n_{t-1} = 0\}$ ,

$$(3.6) \quad \phi_2(\pi_{t\text{-core}}) = \vec{n} = (n_0, n_1, n_2, \dots, n_{t-1}).$$

We call  $\vec{n}$  an  $n$ -vector. It has the following properties.

$$(3.7) \quad \vec{n} \in \mathbb{Z}^t, \quad \vec{n} \cdot \vec{1}_t = 0,$$

and

$$(3.8) \quad |\pi_{t\text{-core}}| = \frac{t}{2} \sum_{i=0}^{t-1} n_i^2 + \sum_{u=0}^{t-1} i n_i,$$

where the  $t$ -dimensional vector  $\vec{1}_t$  has all components equal to 1. The generating function identity that corresponds to this second bijection is

$$(3.9) \quad \sum_{n \geq 0} a_t(n)q^n = \sum_{\substack{\vec{n} \in \mathbb{Z}^t \\ \vec{n} \cdot \vec{1}_t = 0}} q^{\frac{t}{2} \|\vec{n}\|^2 + \vec{b}_t \cdot \vec{n}}.$$

Here

$$(3.10) \quad \|\vec{n}\|^2 = \sum_{i=0}^{t-1} n_i^2, \quad \text{and} \quad \vec{b}_t = (0, 1, 2, \dots, t-1).$$

To construct the  $n$ -vector of  $\pi_{t\text{-core}}$  in (3.6), we follow [8] and define

$$(3.11) \quad \vec{r}(\pi_{t\text{-core}}) = (r_0, r_1, r_2, \dots, r_{t-1}),$$

where for  $0 \leq i \leq t-1$ ,  $r_i(\pi_{t\text{-core}})$  denotes the number of cells labelled  $i \pmod{t}$  in the  $t$ -residue diagram of  $\pi_{t\text{-core}}$ . Then (3.6) can be given explicitly as

$$(3.12) \quad \phi_2(\pi_{t\text{-core}}) = \vec{n} = (r_0 - r_1, r_1 - r_2, r_2 - r_3, \dots, r_{t-1} - r_0).$$

We note that  $\frac{t}{2} \|\vec{n}\|^2$  is a multiple of  $t$  since  $\vec{n} \cdot \vec{1}_t = 0$ . Hence by (3.4) and (3.9) we have

$$(3.13) \quad \sum_{n \geq 0} a_t(tn + \delta)q^{tn+\delta} = \sum_{\substack{\vec{n} \in \mathbb{Z}^t, \vec{n} \cdot \vec{1}_t = 0 \\ \vec{n} \cdot \vec{b}_t \equiv \delta \pmod{t}}} q^{\frac{t}{2} \|\vec{n}\|^2 + \vec{b}_t \cdot \vec{n}},$$

and

$$(3.14) \quad \sum_{n \geq 0} p(tn + \delta)q^n = \frac{1}{(q)_\infty^t} \sum_{n \geq 0} a_t(tn + \delta)q^n,$$

where  $\delta = 0, 1, 2, \dots, t-1$ .

We now assume  $t = 5$ . For the case  $\delta = 4$  the right side of (3.13) can be simplified using the the following change of variables.

$$\begin{aligned}
 n_0 &= \alpha_0 + \alpha_4, \\
 n_1 &= -\alpha_0 + \alpha_1 + \alpha_4, \\
 n_2 &= -\alpha_1 + \alpha_2, \\
 n_3 &= -\alpha_2 + \alpha_3 - \alpha_4, \\
 n_4 &= -\alpha_3 - \alpha_4,
 \end{aligned}
 \tag{3.15}$$

We find  $\vec{n}$  is an  $n$ -vector satisfying  $\vec{n} \cdot \vec{b}_5 \equiv 4 \pmod{5}$  if and only if

$$\vec{\alpha} = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{Z}^5 \tag{3.16}$$

and

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1. \tag{3.17}$$

We call  $\vec{\alpha}$  and  $\alpha$ -vector. Hence, by (3.13) and (3.14) we have

$$\sum_{n \geq 0} a_5(5n+4)q^{n+1} = \sum_{\substack{\vec{\alpha} \cdot \vec{1}_5 = 1 \\ \vec{\alpha} \in \mathbb{Z}^5}} q^{Q(\vec{\alpha})}, \tag{3.18}$$

and

$$\sum_{n \geq 0} p(5n+4)q^{n+1} = \frac{1}{(q)_\infty^5} \sum_{\substack{\vec{\alpha} \cdot \vec{1}_5 = 1 \\ \vec{\alpha} \in \mathbb{Z}^5}} q^{Q(\vec{\alpha})}, \tag{3.19}$$

where

$$Q(\vec{\alpha}) = \|\vec{\alpha}\|^2 - (\alpha_0\alpha_1 + \alpha_1\alpha_2 + \cdots + \alpha_4\alpha_0), \tag{3.20}$$

If  $|\pi| \equiv 4 \pmod{5}$  and  $t = 5$ , we can combine bijections  $\phi_1$  and  $\phi_2$  into a single bijection

$$\Phi(\pi) = (\vec{\alpha}, \vec{\pi}_5), \tag{3.21}$$

such that

$$|\pi| = 5Q(\vec{\alpha}) - 1 + 5 \sum_{i=0}^4 |\hat{\pi}_i|. \tag{3.22}$$

Next, following [9] we define the 5-core crank of  $\pi$  when  $|\pi| \equiv 4 \pmod{5}$  as

$$c_5(\pi) = 1 + \sum_{i=0}^4 i\alpha_i \equiv 2(1 + n_0 - n_1 - n_2 + n_3) \equiv 2 + \sum_{i=-2}^2 ir_{2-i} \pmod{5}, \tag{3.23}$$

where  $\alpha$  is determined by (3.21).

It is easy to check that  $Q(\vec{\alpha})$  in (3.22) remains invariant under the following cyclic permutation

$$\widehat{C}_1(\vec{\alpha}) = (\alpha_4, \alpha_0, \alpha_1, \alpha_2, \alpha_3), \tag{3.24}$$

while  $c_5(\pi)$  increases by 1  $\pmod{5}$  under the map

$$\widehat{O}(\pi) = \Phi^{-1}(\widehat{C}_1(\vec{\alpha}), \vec{\pi}_5). \tag{3.25}$$

In other words, if  $|\pi| \equiv 4 \pmod{5}$ , then

$$|\pi| = |\widehat{O}(\pi)|, \tag{3.26}$$

and

$$(3.27) \quad c_5(\pi) + 1 \equiv c_5(\widehat{O}(\pi)) \pmod{5}.$$

This suggests that all partitions of  $5n + 4$  can be organized into orbits. Each orbit consists of five distinct members:

$$(3.28) \quad \pi, \widehat{O}(\pi), \widehat{O}^2(\pi), \widehat{O}^3(\pi), \widehat{O}^4(\pi),$$

and each element of the orbit has a distinct 5-core crank  $\pmod{5}$ . Clearly, the total number of such orbits is  $\frac{1}{5}p(5n + 4)$ , and so  $p(5n + 4) \equiv 0 \pmod{5}$ . This summarizes the combinatorial proof of (1.1) given in [9]. If we apply the map  $\widehat{O}$  (3.25) to the partitions of  $5n + 4$  that are 5-cores, we find that

$$(3.29) \quad a_5^0(5n + 4) = a_5^1(5n + 4) = \cdots = a_5^4(5n + 4),$$

where, for  $0 \leq j \leq 4$ ,  $a_5^j(n)$  denotes the number of partitions of  $n$  that are 5-cores with 5-core crank congruent to  $j$  modulo 5. Hence,

$$(3.30) \quad a_5^j(5n + 4) = \frac{1}{5}a_5(5n + 4), \quad j = 0, 1, \dots, 4,$$

which proves that

$$(3.31) \quad a_5(5n + 4) \equiv 0 \pmod{5}.$$

Actually, more is true. We have

$$(3.32) \quad a_5(5n + 4) = 5a_5(n).$$

We sketch the combinatorial proof of (3.32) given in [9]. See also [8]. The map  $\theta : P_{5\text{-core}}(n) \longrightarrow P_{5\text{-core}}^0(5n + 4)$ , defined in terms of  $n$ -vectors as

$$(3.33) \quad \begin{aligned} \vec{n} \mapsto \vec{n}' = & (n_1 + 2n_2 + 2n_4 + 1, -n_1 - n_2 + n_3 + n_4 + 1, 2n_1 + n_2 + 2n_3, \\ & -2n_2 - 2n_3 - n_4 - 1, -2n_1 - n_3 - 2n_4 - 1), \end{aligned}$$

is a bijection. Here  $P_{5\text{-core}}(n)$  is the set of all 5-cores of  $n$ , and  $P_{5\text{-core}}^0(n)$  is set of all 5-cores of  $n$  with 5-core crank congruent to zero modulo 5. Since  $\theta$  is a bijection, we have

$$(3.34) \quad a_5(n) = a_5^0(5n + 4).$$

The proof of (3.32) easily follows from (3.30) and (3.34). Finally, we remark that Ramanujan's result (1.7) is a straightforward consequence of (3.14) with  $(t, \delta) = (5, 4)$ , (3.32), and (3.5) with  $t = 5$ .

#### 4. REFINEMENT OF RAMANUJAN'S MOD 5 CONGRUENCE, THE SRANK AND THE 5-CORE CRANK

In the previous section we discussed the combinatorial proof in [9] of Ramanujan's congruence (1.1) using the the 5-core crank (3.23). It is somewhat unexpected that the 5-core crank can be employed to prove the refinement (1.21) as well.

In fact, we were amazed to discover the following elegant formulas

$$(4.1) \quad \text{srnk}(\pi_{5\text{-core}}) \equiv \sum_{i=0}^4 (n_i + i)^3 \pmod{4},$$

$$(4.2) \quad \begin{aligned} \text{srnk}(\pi) &\equiv \text{srnk}(\pi_{5\text{-core}}) + \sum_{i=0}^4 \text{srnk}(\widehat{\pi}_i) \\ &\quad + 2 \sum_{i=0}^4 |\widehat{\pi}_i|(n_i + i) \pmod{4}, \end{aligned}$$

where  $\pi_{5\text{-core}}, \vec{\pi} = (\widehat{\pi}_0, \widehat{\pi}_1, \widehat{\pi}_2, \widehat{\pi}_3, \widehat{\pi}_4)$  are determined by (3.1) with  $t = 5$ , and

$$\vec{n} = (n_0, n_1, \dots, n_4) = \phi_2(\pi_{5\text{-core}}).$$

In spite of their simple appearance, the above formulas are far from obvious. The proof of (4.1)-(4.2) will be given elsewhere. Here we restrict our attention to some implications of (4.1)-(4.2).

First, we note that if  $|\pi_{5\text{-core}}| \equiv 4 \pmod{5}$ , then (4.1) can be written in terms an  $\alpha$ -vector (3.15) as

$$(4.3) \quad \text{srnk}(\pi_{5\text{-core}}) \equiv \alpha_0\alpha_1(\alpha_0 - \alpha_1) + \alpha_1\alpha_2(\alpha_1 - \alpha_2) + \dots + \alpha_4\alpha_0(\alpha_4 - \alpha_0) \pmod{4}.$$

Similarly, if  $|\pi| \equiv 4 \pmod{5}$ , then

$$(4.4) \quad \begin{aligned} \text{srnk}(\pi) &\equiv \alpha_0\alpha_1(\alpha_0 - \alpha_1) + \dots + \alpha_4\alpha_0(\alpha_4 - \alpha_0) + \sum_{i=0}^4 \text{srnk}(\widehat{\pi}_i) \\ &\quad + 2\{(\alpha_0 + \alpha_4)|\widehat{\pi}_0| + (\alpha_2 + \alpha_3)|\widehat{\pi}_1| + (\alpha_1 + \alpha_2)|\widehat{\pi}_2| \\ &\quad + (\alpha_0 + \alpha_1)|\widehat{\pi}_3| + (\alpha_3 + \alpha_4)|\widehat{\pi}_4|\} \pmod{4}. \end{aligned}$$

Remarkably, (4.3) suggests that  $\text{srnk}(\pi_{5\text{-core}})$  with  $|\pi_{5\text{-core}}| \equiv 4 \pmod{5}$  remains invariant mod 4 under the cyclic permutation (3.24), and we have the following refinement of (3.30):

$$(4.5) \quad a_{5,i}^j(5n+4) = \frac{1}{5}a_{5,i}(5n+4),$$

where  $j = 0, \dots, 4$  and  $i = 0, 2$ . Here  $a_{5,i}(n)$  denotes the number of 5-cores of  $n$  with  $\text{srnk} \equiv i \pmod{4}$ , and  $a_{5,i}^j(n)$  denotes the number of 5-cores of  $n$  with  $\text{srnk} \equiv i \pmod{4}$  and 5-core  $\text{crank} \equiv j \pmod{5}$ . Moreover, it is not difficult to verify that the map  $\theta$ , given by (3.33), preserves the  $\text{srnk} \pmod{4}$ . Indeed, recalling that  $n_0 + n_1 + n_2 + n_3 + n_4 = 0$  we find after some simplification that

$$(4.6) \quad \begin{aligned} \sum_{i=0}^4 ((n_i + i)^3 - (n'_i + i)^3) &\equiv 2(n_0n_2(n_0 + n_2) + n_1n_3(n_1 + n_3) + n_2n_3(n_2 + n_3) \\ &\quad + n_1(n_1 + 1) + n_2(n_2 + 1) + n_3(n_3 + 1)) \\ &\equiv 0 \pmod{4}, \end{aligned}$$

where  $\vec{n}'$  is defined in (3.33). Hence, (3.34) and (3.32) can be refined as

$$(4.7) \quad a_{5,i}(n) = a_{5,i}^0(5n+4), \quad (i = 0, 2),$$

and

$$(4.8) \quad a_{5,i}(5n+4) = 5a_{5,i}(n), \quad (i = 0, 2),$$

respectively.

It is less trivial to prove the 5-core crank analogue of Theorem 2.2. Namely,

**Theorem 4.1.** *The residue of the 5-core crank mod 5 divides the partitions enumerated by  $p_i(5n+4)$  with  $i = 0, 2$  into five equal classes.*

*Proof.* We sketch a proof assuming (4.4) and (4.5) hold. We define the cyclic shift operator  $\widehat{C}_2$  by

$$(4.9) \quad \widehat{C}_2(\vec{\pi}_5) = (\widehat{\pi}_4, \widehat{\pi}_2, \widehat{\pi}_3, \widehat{\pi}_0, \widehat{\pi}_1),$$

Next, we use (4.9) to modify (3.25) as

$$(4.10) \quad \widehat{O}_s(\pi) = \Phi^{-1}(\widehat{C}_1(\vec{\alpha}), \widehat{C}_2(\vec{\pi}_5)),$$

where  $\Phi(\pi) = (\vec{\alpha}, \vec{\pi}_5)$ . Fix  $i = 0, 2$ . By (4.4) we see that  $\widehat{O}_s$  preserves the srnk mod 4, and we may assemble all partitions of  $5n+4$  with  $\text{srnk} \equiv i \pmod{4}$  into orbits:

$$\pi, \widehat{O}_s(\pi), \widehat{O}_s^2(\pi), \widehat{O}_s^3(\pi), \widehat{O}_s^4(\pi),$$

where  $\pi$  is some partition of  $5n+4$  with  $\text{srnk}(\pi) \equiv i \pmod{4}$ . As before, each orbit contains exactly five members and the 5-core crank increases by 1 mod 5 along the orbit. The number of these orbits is  $\frac{1}{5}p_i(5n+4)$ , consequently  $p_i(5n+4) \equiv 0 \pmod{5}$  and the result follows.  $\square$

Theorem 4.1 is illustrated below in Table 2, which contains all 30 partitions of 9, organized into 6 orbits. Each row in this table represents an orbit, and the first row lists all partitions of 9 that are 5-cores.

TABLE 2.

	5-core crank $\equiv 0 \pmod{5}$	1(mod 5)	2(mod 5)	3(mod 5)	4(mod 5)
srnk $\equiv 0$ (mod 4)	$(1^4, 5^1)$	$(1^3, 3^2)$	$(1^4, 2^1, 3^1)$	$(1^1, 2^1, 6^1)$	$(2^2, 5^1)$
	$(1^5, 2^2)$	$(2^3, 3^1)$	$(1^2, 7^1)$	$(4^1, 5^1)$	$(1^3, 2^1, 4^1)$
	$(3^3)$	$(1^9)$	$(1^1, 3^1, 5^1)$	$(1^2, 2^2, 3^1)$	$(9^1)$
	$(2^1, 7^1)$	$(1^2, 2^1, 5^1)$	$(1^1, 2^4)$	$(1^6, 3^1)$	$(1^1, 4^2)$
srnk $\equiv 2$ (mod 4)	$(1^3, 2^3)$	$(1^3, 6^1)$	$(2^1, 3^1, 4^1)$	$(1^1, 8^1)$	$(1^2, 3^1, 4^1)$
	$(3^1, 6^1)$	$(1^1, 2^2, 4^1)$	$(1^7, 2^1)$	$(1^1, 2^1, 3^2)$	$(1^5, 4^1)$

New, we state some new formulas for  $a_{5,0}(n)$ :

$$(4.11) \quad a_{5,0}(4n) = a_5(4n),$$

$$(4.12) \quad a_{5,0}(4n+1) = a_5(4n+1),$$

$$(4.13) \quad a_{5,0}(4n+2) = 0,$$

$$(4.14) \quad a_{5,0}(4n+3) = a_5(n).$$

Formulas (4.11)-(4.13) follow from (4.1). Formula (4.14) is a consequence of the following bijective map, defined in terms of  $n$ -vectors by

$$(4.15) \quad \vec{n} \mapsto \vec{n}' = (2n_1, 1 + 2n_4, 2n_2, -1 + 2n_0, 2n_3).$$

The important properties of (4.15) are

$$(4.16) \quad |\phi_2^{-1}(\vec{n}')| = 4|\phi_2^{-1}(\vec{n})| + 3,$$

and

$$(4.17) \quad \text{srnk}(\phi_2^{-1}(\vec{n}')) \equiv 0 \pmod{4}.$$

The details will be given elsewhere.

## 5. CONCLUDING REMARKS

While the strank development in section 2 followed naturally from the Andrews product (1.14), the 5-core crank development in the previous section arose in an unexpected fashion. One may wonder, if there exists an additional new partition statistic, closely related to the Dyson rank, whose residue mod 5 splits the partitions enumerated by  $p_i(5n+4)$  with  $i = 0, 2$  into 5 equal classes. Finally, we would like to pose the

**Problem.** Is there an analogue of the Stanley rank, which gives a refinement for Ramanujan's partitions congruences mod 7 and mod 11?

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